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Directed polymers with killing

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Abstract. The problem of directed polymers in a random medium is generalized to include a killing probability. The model is then studied using results of the theory of extreme statistics. These reveal that, in a region of parameters, the thermodynamics of polymers with killing in finite dimensions is similar to that of the mean field solution of directed polymers on a Cayley tree. The implication of these results for the behaviour of directed polymers in high dimensions is also discussed.

1. Introduction

Directed polymers in a random medium (DPRM) [1, 2] have been the focus of great recent interest. On the one hand it is a prototype model for a system with quenched disorder [3, 4]. On the other hand, invoking universality, the behaviour of directed polymers has been related to a number of other problems of great interest [6–8]. Over the last decade, a satisfactory understanding of the model in $d = 1$ [2] and $d = \infty$ dimensions [3–5] has been reached. The understanding of DPRM in intermediate dimensions is based mainly on numerical simulations [9, 10]. In particular, the strong coupling phase of the model has been shown to be beyond the range of standard theoretical approaches [2, 11], because of the failure of perturbation expansions. One still open question [1, 11–16] concerns the existence of an upper critical dimension, above which the exponents stick to their $d = \infty$ values.

We shall consider directed polymers from the point of view of the theory of extreme statistics [17]. Since the thermodynamics of DPRM is controlled by a zero temperature fixed point, the problem essentially consists in the characterization of the statistics of the ground state energy, i.e. of the directed path with the minimum energy. The theory of extreme statistics [17] appears as a natural candidate to address this question.

The next section shows that, under the assumption of independent and identically distributed (IID) walk energies, the prediction of the theory of extreme statistics reproduces almost exactly the result found on the Cayley tree. This suggests a very intuitive understanding of the behaviour of DPRM for $d \geqslant d_c$: the ensemble of walks has the same properties of non-interacting (i.e. non-intersecting) walks. In section 2.2 the correlation among walks is taken into account. This turns out to be too strong in order to recover the IID result. We turn, in section 3, to a generalized model of directed polymers, where a killing probability is introduced. The latter is tuned in such a way as to eliminate mainly high energy walks and, at the same time, to satisfy the hypothesis of the theory of extreme statistics in the presence of correlation. The results obtained, which apply to a region in

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the parameter space where the set of surviving walks is weakly correlated, reveals that this model, in finite dimension $d > 2$, has a phase transition to a low temperature phase which has the same characteristics of the mean field solution ($d = \infty$) of DPRM.

In the final section we shall try to extract some information on the original problem of DPRM from our results on DPRM with killing. It will be argued that our results are not relevant for the low temperature phase. They yield upper bounds for the ground state energy and for the transition temperature. However, our model also provides a connection to directed percolation, which to some extent describes the organization of the disorder. Exploiting this relation and the results of the previous sections, we will present arguments which suggest a finite upper critical dimension $d_c = 4$.

2. Directed walks in random media and extreme statistics

The model of DPRM is defined on a Euclidean lattice. On each site i a random energy $\underline{\epsilon}_i$ is defined, which is drawn from a distribution density $\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{Prob}(\underline{\epsilon}_i \leqslant x) \equiv \phi(x)$ (for clarity, we shall underline the symbols when they refer to random variables). Consider all the directed walks with a fixed origin and of length L. Directed means that each step of the walk has a positive component in the 'time' direction, so that the walk w passes through the sites $i_w(t) = (r_w(t), t)$ where $t = 0, \ldots, L$ and $r_w(0) = 0$. If the coordination number of the lattice in the 'space' sub-lattice is z, there will be $N = z^L$ distinct walks. The energy of a walk is the sum of the random energies of the sites it passes through: $\underline{E}_w = \sum_t \underline{\epsilon}_{i_w(t)}$. Introducing a temperature T, one can now build a partition function $Z(T, L)$ and all the thermodynamic quantities which are related to it. For dimensions $d > 2$ there is a phase transition between an entropy dominated phase, for $T > T_c$, and an energy dominated one for $T < T_c$. In the latter phase Z is dominated by the walk with the smallest energy $\underline{E}_< = \min_w\{\underline{E}_w\}$, hereafter called the ground state (GS). The problem is to characterize the L dependence of $\underline{E}_<(L)$:

$$\underline{E}_<(L) = \alpha L + bL^{\Delta-1} + cL^{\omega}\underline{\Lambda}. \tag{1}$$

where the first term is the extensive part of $\underline{E}_<(L)$, the second is a scaling correction, and the third describes the sample to sample fluctuations of $\underline{E}_< < (L)$ ($\underline{\Lambda}$ is an L-independent random variable). The behaviour of the amplitude of the fluctuations of $\underline{E}_<(L)$ with L, defines the exponent ω, which is the quantity of interest one would like to calculate as a function of the space dimensionality d. The exponent ω is known to be $\frac{1}{3}$ in $d = 1 + 1$, and to be zero in the limit $d \to \infty$. The first terms of the $1/d$ expansion [14] as well as other recent investigations [12, 16, 11], suggest that $\omega(d)$ vanishes at a finite dimension d_c and then stays zero for $d > d_c$.

2.1. Independent walks

In this section, we define a 'walk' w as a collection of L sites i_w. The energy $\underline{E}_w$ of the walk, is the sum of L independent 'site' energies $\underline{\epsilon}_{i_w}$, and it is itself an independent variable for each walk. The distribution density of $\underline{E}_w$ is obtained in the usual way:

$$p(E) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \langle \mathrm{e}^{-\mathrm{i}kx}\rangle^L \mathrm{e}^{\mathrm{i}kE} \tag{2}$$

where here and in the following $\langle\cdot\rangle = \int_{-\infty}^{\infty} \phi(x)(\cdot)\,\mathrm{d}x$ and we consider zero average and unit variance site energies: $\langle\underline{\epsilon}\rangle = 0$, $\langle\underline{\epsilon}^2\rangle = 1$.

The distribution of $\underline{E}_< = \min(\underline{E}_1, \ldots, \underline{E}_N)$, for $N = z^L$ independent variables is obtained in the following way. The probability that $\underline{E}_< \geqslant x$, which is $1 - P_<(x)$, is the probability that all the $\underline{E}_w \geqslant x$, i.e. $P_<(x) = 1 - [1 - P(x)]^N$. To obtain a finite limit we have to shift and scale $\underline{E}_<$ in an appropriate way, $\underline{E}_< = a_L + b_L \underline{\Lambda}$, so that the distribution of $\underline{\Lambda}$ is non-degenerate and independent of L. The application of these arguments to our case is discussed in detail in appendix A. The result is that

$$\underline{E}_< = \alpha L + \frac{1}{\gamma}\left[\frac{1}{2}\ln(2\pi\sigma^2\gamma^2 L) + \underline{\Lambda}\right] \tag{3}$$

where $\underline{\Lambda}$ is a random variable with distribution density

$$\lambda(x) = \exp[x - \exp(x)]; \tag{4}$$

α and γ are given by equations

$$\alpha = \frac{\langle x \mathrm{e}^{-\gamma x}\rangle}{\langle \mathrm{e}^{-\gamma x}\rangle} = -\frac{1}{\gamma}\ln(z\langle \mathrm{e}^{-\gamma x}\rangle). \tag{5}$$

and

$$\sigma^2 = \frac{\langle x^2 \mathrm{e}^{-\gamma x}\rangle}{\langle \mathrm{e}^{-\gamma x}\rangle} - \left(\frac{\langle x \mathrm{e}^{-\gamma x}\rangle}{\langle \mathrm{e}^{-\gamma x}\rangle}\right)^2. \tag{6}$$

This result (called the IID result for short, in the following) requires, as seen in the appendix, that $\phi(x)$ has finite moments of all orders.

In the limit $L \to \infty$ the central limit theorem would suggest that $\underline{E}_w$ are Gaussian variables. However, the limit involved in the theory of extreme statistics, probes the tails of the distribution of $\underline{E}_w$, where the central limit theorem does not hold. This is why, for power law distributions $\phi(x) \sim x^{-\mu-1}$, even if the second moment of $\phi(x)$ is finite, one can expect an anomalous behaviour with the exponent ω depending on μ even for $\mu > 2$, as seen in $d = 1 + 1$ simulations [18]. Note that the distribution $\lambda(x)$ is highly asymmetric in the tails. The exponential decay for $x \to -\infty$ agrees with the argument of [19] for $\omega = 0$, while for $x \to \infty$ it drops to zero much faster. A similar, less pronounced, asymmetry was found in finite dimensions [9].

Comparing equation (3) with equation (1) one concludes that $\omega = 0$. The expression for the energy density α coincides with that obtained for the Cayley tree. The expression of the coefficient of the random part also coincides, whereas that of the logarithmic part differs by a factor of 3. This suggests that on the Cayley tree walk energies cannot be considered as independent random variables either. Under the assumption of independence, the model is essentially a version of the random energy model [20], studied in the context of spin glasses.

2.2. Correlated walks

A simple observation, which casts serious doubts on the possible relevance of this result to DPRM in any dimension, is that the z^L variables $\underline{E}_w$ are built using only L^{d+1} independent variables $\underline{\epsilon}_r$. One can, however, resort to the following result of the theory of extreme statistics [17]: consider the set $\underline{C}_L$ of all the pairs (w, v) of indices such that $\underline{E}_w$ and $\underline{E}_v$ are dependent. If (i) the number of correlated pairs, $|\underline{C}_L|$ is a negligible fraction of the total number of pairs, and (ii) if, for $U_L = a_L + b_L x$, with a_L and b_L given by the expressions for independent walks,

$$\lim_{L\to\infty} \sum_{(w,v)\in\underline{C}_L} P(\underline{E}_w < U_L; \underline{E}_v < U_L) = 0 \tag{7}$$

then $\underline{E}_< = \min\{\underline{E}_1, \ldots, \underline{E}_n\} = a_L + b_L \underline{\Lambda}$, with a_L, b_L and the distribution of $\underline{\Lambda}$ given by the IID result of the previous section. Therefore, *only a global control of pairwise correlation in the tail of the distribution is needed for the IID result to hold*. Moreover, one needs to consider only the marginal distributions of one walk and of a pair of walks.

The correlation $\langle \underline{E}_w \underline{E}_v \rangle = m_{w,v}$ between two walk energies, assuming that $\langle \underline{\epsilon}^2 \rangle = 1$, is equal to the number $m_{w,v}$ of sites on which these intersect [21]. In $d \leqslant 2$ the number of intersections of two random walks diverges with their length, while above two dimensions $m_{w,v}$ is almost surely finite. Therefore, $\underline{C}_L$, the set of pairs of walks which intersect at least in one point, essentially coincides with the set of all pairs for $d \leqslant 2$. We can make the number of correlated pairs $|\underline{C}_L|$ small with respect to the total number of walks, and therefore satisfy condition (i), only in $d > 2$. This is achieved by neglecting the contribution to the energy $\underline{E}_w$ of the first ℓ steps. Out of all the pairs of walks, only those which intersect after ℓ steps have correlated energies. If $d > 2$, it is enough to take $\ell \sim \ln L$, which is negligible as compared to L as $L \to \infty$, in order to satisfy the first condition. Therefore for $d > 2$ the first condition can be satisfied by an appropriate adjustment of sub-leading terms of order $\ln L$.

The second condition, equation (7), can be checked by a direct calculation as explained in appendix B. The condition under which the IID result holds, is that

$$\Pi \langle \mathrm{e}^{-2\gamma \underline{\epsilon}} \rangle \leqslant \langle \mathrm{e}^{-\gamma \underline{\epsilon}} \rangle^2. \tag{8}$$

Here γ is the parameter defined in the previous section and Π is the return probability of a walk on the d-dimensional lattice. This yields the probability $Q_m \propto \Pi^m$ that two walks have m intersections.

For example, for normal site energies, for which $\alpha = -\sqrt{2 \ln z}$, the inequality (8) reads (see appendix B) $\Pi z^2 \leqslant 1$. Unfortunately this does not hold for any Euclidean lattice, where $\Pi z \geqslant 1$. It also fails to apply to the Cayley tree, which has a scaling limit very close to the result for IID variables. This suggests that one needs a further ingredient to apply these results of extreme order statistics to the DPRM. The key observation is that in the saddle point evaluation of equation (7) one finds a term which depends only on the number of walks and another term depends only on correlation (i.e. on Π). It is impossible to decrease the correlation between walks on a given lattice, but it is however very easy to decrease their number, i.e. by annihilating them.

3. Walks with killing

Let us introduce a killing probability $1 - p$ in the DPRM model: a walk is killed with probability $1 - p$ on each site and otherwise it survives. We shall allow the probability p to depend on the value $\underline{\epsilon}_i$ of the energy on site i.

Let $\mathcal{S}_i$ be the set of walks arriving at site $i = (\boldsymbol{x}, t)$, which survive all killing attempts. We shall frequently denote by $\mathcal{S}$ the set of all walks of length L which survive the killing process. Before analysing the extreme statistics of this model let us mention that a partition function can be introduced also in this model:

$$Z_i(\beta) = \sum_{w \in \mathcal{S}_i} \mathrm{e}^{-\beta \underline{E}_w}. \tag{9}$$

The usual recursion relation for DPRM, however, cannot be generalized to walks with killing. Indeed writing explicitly Z_i for a walk of $t+1$ steps ($i = (\boldsymbol{x}, t+1)$) one finds:

$$Z_{(\boldsymbol{x},t+1)}(\beta) = \mathrm{e}^{-\beta \underline{\epsilon}(\boldsymbol{x},t+1)} \sum_{(\boldsymbol{y},\boldsymbol{x})} \sum_{w \in \mathcal{S}_(\boldsymbol{y},t)} \underline{\kappa}_{w,t} \mathrm{e}^{-\beta \underline{E}_w}. \tag{10}$$

Here the first sum runs over the z neighbour sites of $\boldsymbol{x}$ and the second over all the walks which survive up to $(\boldsymbol{y}, t)$. The random variable $\underline{\kappa}_{w,t}$ takes the value 1 with probability $p(\underline{\epsilon}_{(\boldsymbol{x},t+1)})$ and is 0 otherwise. Since $\underline{\kappa}_{w,t}$ depends on the walk's index w, the right-hand side cannot be expressed in terms of $Z_{(\boldsymbol{y},t)}$. Appendix C analyses the recurrence relation which results from taking the average over the variables $\underline{\kappa}$ in equation (10).

Killing reduces the number of walks. The probability that the walk w survives is given by

$$P_s(w) = \prod_{i=1}^{L} p(\underline{\epsilon}_{i_w}). \tag{11}$$

The number $\underline{n}$ of surviving walks is a random number. Its average is equal to $[z\langle p(\underline{\epsilon})\rangle]^L$, where $\langle p(\underline{\epsilon})\rangle$ is the average probability of surviving. In order to have a macroscopic number of walks, so that taking the minimum has some sense, we shall require

$$z\langle p(\underline{\epsilon})\rangle \geqslant 1. \tag{12}$$

For correlated walks, as we shall see, we need an additional condition to be sure that $\underline{n}$ is macroscopic.

3.1. Independent walks with killing

Killing introduces an additional source of randomness, related to the particular realization of the set $\mathcal{S}$ of surviving walks. For each realization of $\mathcal{S}$, under the assumption of independence, one should consider the limit $L \to \infty$ as done in appendix A. The problem is to find the constants a_L and b_L such that

$$\lim_{L\to\infty} \sum_{w\in\mathcal{S}} P(\underline{E}_w < a_L + b_L x) = c(x)$$

If $\underline{n}$ is macroscopic, i.e. exponentially large in L, one can assume that almost all realizations of $\mathcal{S}$ will yield the same result. Therefore one can calculate the coefficients a_L and b_L for which the average of the above equation is finite. Apart from this, the calculation of a_L and b_L follows exactly the same lines of appendix A, provided that a factor $p(x)$ is inserted in each average. More precisely one finds that equation (3) holds with α and γ given by

$$\alpha = \frac{\langle x p(x) \mathrm{e}^{-\gamma x}\rangle}{\langle p(x)\mathrm{e}^{-\gamma x}\rangle} = -\frac{1}{\gamma}\ln[z\langle p(x)\mathrm{e}^{-\gamma x}\rangle]. \tag{13}$$

The same result can be reached by considering one typical realization of $\mathcal{S}$. The number of walks in a typical sample is $n \simeq (z\langle p(\underline{\epsilon})\rangle)^L$. The effective site energy of sites on which the walks survive is

$$\tilde{\phi}(x) = \frac{1}{\langle p(\underline{\epsilon})\rangle} p(x)\phi(x). \tag{14}$$

Therefore one can repeat the derivation of appendix A and recover equations (13).

3.2. Correlated walks with killing

The first effect of correlation is to make the stochastic nature of the variable $\underline{n}$ slightly more complex. It can be shown, following standard methods [22], that a sufficient condition for $\underline{n}$ to attain a macroscopic value with large probability, is that

$$\tilde{\Pi} \equiv \Pi \frac{\langle p^2(\underline{\epsilon})\rangle}{\langle p(\underline{\epsilon})\rangle^2} < 1. \tag{15}$$

The argument goes roughly as follows. The variable $\underline{W} = \underline{n}/(z\langle p(\underline{\epsilon})\rangle)^L$ has a limit distribution when $L \to \infty$. All one needs to show is that this limit distribution is not degenerate, i.e. $p(W) \neq \delta(W)$. In order to show this it is enough to show that the second moment of $\underline{W}$ is finite, which leads to the condition (15).

Equation (15) can also be understood more simply by observing that the left-hand side is the probability $\tilde{\Pi}$ that two walks in $\mathcal{S}$ intersect, renormalized by the effect of killing. Consider indeed two walks w and v. The probability that each of them belongs to $\mathcal{S}$ is given by equation (11), which is on average $\langle p(\underline{\epsilon})\rangle^L$. However, if w and v intersect on m sites, the probability that both belong to $\mathcal{S}$ is not $\langle p(\underline{\epsilon})\rangle^{2L}$ but rather $\langle p(\underline{\epsilon})\rangle^{2(L-m)}\langle p^2(\underline{\epsilon})\rangle^m$. Therefore the probability that two walks intersect on m sites and belong to $\mathcal{S}$ can be expressed as the probability that each of them belongs to $\mathcal{S}$, which is $\langle p(\underline{\epsilon})\rangle^{2L}$, times the effective probability of m intersections $\tilde{\Pi}^m$. Clearly one needs $\tilde{\Pi} < 1$ in order to have a normalizable distribution of m. $\tilde{\Pi} \geqslant 1$ means that the walks in $\mathcal{S}$, if any, are highly correlated. Note that, since $\langle p^2(\underline{\epsilon})\rangle \geqslant \langle p(\underline{\epsilon})\rangle^2$, killing in general increases the correlation between walks.

The necessary condition required to recover the IID result is

$$\lim_{L\to\infty} \sum_{(w,v)\in \underline{C}_L(\mathcal{S})} p_{(w,v)}(a_L + b_L x, a_L + b_L x) = 0 \tag{16}$$

where $\underline{C}_L(\mathcal{S})$ is the set of pairs (w, v) of indices of walks for which the number of intersections $m_{w,v} \geqslant 1$. We will take the average of equation (16) over the realizations of $\mathcal{S}$, assuming that the result is self-averaging. In doing this, one incurs the risk that the limit is dominated by rare realizations of $\mathcal{S}$ for which equation (16) is exceptionally large. However, if the average over realizations of equation (16) vanishes, it surely vanishes for the typical realization.

In order to evaluate the average of equation (16) one writes the sum over pairs $(w, v) \in \underline{C}_L(\mathcal{S})$ as a sum over all pairs times an indicator function $I_{(w,v)}$ which is 1 if both walks survive and 0 otherwise. The average can then be exchanged with the summation and one is left with the evaluation of $\overline{I_{(w,v)} p_{(w,v)}(a_L + b_L x, a_L + b_L x)}$. This is simply the probability that two walks have energies $a_L + b_L x$ and that they both survive, and it is easily evaluated. It is then straightforward to apply the considerations of section 2.2 and appendix B and to arrive at the condition

$$\Pi \frac{\langle p^2(\underline{\epsilon}) \mathrm{e}^{-2\gamma\underline{\epsilon}}\rangle}{\langle p(\underline{\epsilon})\mathrm{e}^{-\gamma\underline{\epsilon}}\rangle^2} \leqslant 1 \tag{17}$$

for the validity of the IID result. Here γ is given, together with α by equations (13).

The same result can be found by considering a typical realization of $\mathcal{S}$ in which $n = [z\langle p(\underline{\epsilon})\rangle]^L$ walks survive. The pairwise correlation between these walks is described by $\tilde{\Pi}$ as given in equation (15). Finally one needs to consider an effective distribution for the site energies. However in this case equation (14) does not apply to the sites where the walks intersect. On these sites, indeed, the effective distribution has to account for the fact that both walks survive, i.e. $\tilde{\phi}(x) = p^2(x)\phi(x)/\langle p^2(x)\rangle$. The considerations of appendix B lead easily to the above condition. Note that if (17) holds with $\gamma > 0$, then $\tilde{\Pi} < 1$, which implies that $\underline{n} \sim [z\langle p(\underline{\epsilon})\rangle]^L$ is indeed macroscopic.

3.3. Results

In order to illustrate the procedure and to draw conclusions we need to specify the model. This is defined by:

(i) the site energy probability distribution $\phi(x)$;

(ii) the geometry of the lattice, which yields $z(d)$ and $\Pi(d)$ as function of the space dimensionality;

(iii) the killing probability $p(x)$.

Let us focus on standard normal site energies and consider walks in the $\hat{t} = (1, 0, \ldots, 0)$ direction of a hypercubic lattice with $\boldsymbol{r}_w(t+1) = \boldsymbol{r}_w(t) + \hat{t} \pm \hat{x}_i$ with $i = 1, \ldots, d$. Therefore

$$z = 2d \qquad \text{and} \qquad \Pi = 1 - \left\{ \int_{-\pi}^{\pi} \frac{\mathrm{d}^d k}{(2\pi)^d} \left[1 - \frac{1}{d} \sum_{i=1}^{d} \cos k_i \right]^{-1} \right\}^{-1}. \tag{18}$$

Since we are interested in the statistics of low energy walks without killing, we shall use the function $p(x)$ in a variational way. In practice we shall consider a family of functions $\{p(x)\}$ and calculate, if possible, the GS energy $\alpha[p(x)]$ for each $p(x)$. Finally we shall look for the function $p(x)$ which minimizes the GS energy. The following:

$$\alpha_{\min} = \min_{p(x)} \alpha[p(x)] \tag{19}$$

will be our best estimate (upper bound) of the GS energy of DPRM.

In order to find a suitable set of functions we observe that $p(x)$ should eliminate mainly the high energy walks. The simplest way of achieving this is to take $p(x) = 1$ for $x \leqslant \tau$. In this way walks can be killed only when they step on sites with energy bigger than τ. If one could take τ big enough so that no site of the best path has an energy bigger than τ, one would be sure that the GS walk is in the set $\mathcal{S}$. We shall see that this is not possible in principle, since it is possible to find very large site energies on the best path. Since the minimization applies to the sum of site energies, we still have a probability that low energy walks, and eventually the GS walk, gets killed during one of its rare passages on sites of energy bigger than τ. Typically the smallest among the surviving walks will have a bigger energy than the global GS walk. A second important feature is how the killing probability depends on the energy of site i for $\underline{\epsilon}_i > \tau$. The simplest choice is to take a Boltzmann factor-like surviving probability

$$p_{\eta,\tau}(x) = \theta(\tau - x) + \mathrm{e}^{-\eta(x-\tau)}\theta(x - \tau). \tag{20}$$

With this choice, the surviving probability of a walk will depend on the total energy of sites above the threshold: $P_{\mathrm{s}}(w) = \mathrm{e}^{-\eta E_{>}(w)}$ where $E_{>}(w) = \sum_i (\underline{\epsilon}_i^{(w)} - \tau)\theta(\underline{\epsilon}_i^{(w)} - \tau)$. The parameters in $p(x)$ will be tuned in a variational way, in order to find the values for which the condition (17) holds and the GS energy per site α is minimum. One could easily think of more complex $p(x)$. $p(x)$ in equation (20) seems to be the natural choice in our problem where the minimization involves the energy. We indeed found that, of all the $p(x)$ we considered, this gave the minimum energy. It is very easy to find the equations which yield α and γ. However, since they are implicit equations and their form is not particularly illuminating, we refrain from displaying them.

Let us discuss the results as a function of η. The region of parameters (η, τ) where $z\langle p(\underline{\epsilon})\rangle \geqslant 1$ and the condition (17) holds, for a given dimension $d > 2$, is schematically depicted in figure 1. Below this region, if τ is too small, there are no surviving walks, i.e. $z\langle p(\underline{\epsilon})\rangle < 1$. Above this region, instead, the walks are too strongly correlated and the condition (17) does not hold. Technically, the saddle point calculation of equation (16) shows that the sum is dominated by pairs (w, v) of walks with an 'extensive' number of intersections $m_{w,v} \propto L$. This signals the need for different normalization coefficients, a_L and b_L.

In the shaded region of figure 1, the minimum value of α, for fixed η, is obtained when the inequality (17) is saturated, i.e. for the maximum value of τ (indeed the left-hand side of the inequality (17) is an increasing function of τ, while α, for fixed η, is a decreasing

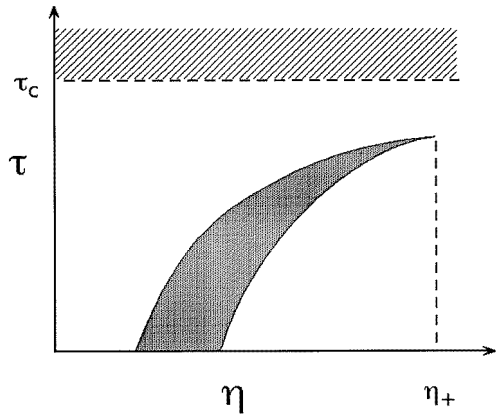

Figure 1. Schematic 'phase diagram' for walks with killing. Below the shaded region the number of surviving walks is zero. In the shaded region the results discussed in the text apply. The upper boundary of the shaded region is where (14) turns into an equality. Above the line $\tau = \tau_c$ the sites where $p(\underline{\epsilon}) = 1$ percolate through the lattice.

function of τ). Since we are looking for the minimum α, we shall fix $\tau(\eta)$ as a function of η, in such a way that (17) becomes an equality. In this case, the arguments of [21] show that γ is the inverse critical temperature for the set of walks with killing. As shown in appendix C, the entropy vanishes at the transition and remains zero in the whole low temperature phase. This situation is similar to the solution of DPRM on the Cayley tree [3]. Indeed, as for directed polymers on a Cayley tree, the low temperature phase is completely frozen.

For $\eta = 0$ we can apply the IID result if we take $p_{\eta,\tau}(x) \to p$ as $\eta \to 0$. This implies $\tau \to -\infty$. One can easily find that, in this case

$$\lim_{\eta\to 0} \gamma = -\lim_{\eta\to 0} \alpha = \sqrt{-\ln \Pi}. \tag{21}$$

In this limit we recover the upper bound to the critical temperature obtained in [21].

As η increases both γ and α decrease initially. The behaviour is sketched in figure 2 for $d = 4$ and 6 (note that both the energies and η are scaled to the IID result by a factor $\sqrt{2 \ln z}$). For high enough dimensions $d > d_0$, $\alpha(\eta)$ reaches at some point a minimum $\alpha_{\min}$ and then starts to increase. Here d_0 is a dimension which depends on the structure of the lattice through z and Π. For $d < d_0$ the increasing branch of $\alpha(\eta)$ is absent. This is because the IID result applies as long as both the inequalities (17) and $z\langle p(\underline{\epsilon})\rangle \leqslant 1$ can be satisfied. There is a maximum value of $\eta \equiv \eta_+$ above which this will no more be the case which is given by saturating both inequalities. It is straightforward to check that the equation for η_+ and $\tau(\eta_+) = \tau_+$ is

$$z\langle p_{\eta_+,\tau_+}(\underline{\epsilon})\rangle = z^2 \Pi \langle p_{2\eta_+,\tau_+}(\underline{\epsilon})\rangle = 1. \tag{22}$$

Note that for the Cayley tree $\Pi = 1/z$, therefore $\eta_+ = \infty$ and $z\langle p_{\infty,\tau_\infty}(\underline{\epsilon})\rangle = 1$. For any Euclidean lattice the equation (22) will be satisfied for a finite $\eta_+ < \infty$. Note also that, as $\eta \to \eta_+$, $\gamma \to 0$. This implies on one hand that in the set $\mathcal{S}$ all walks have energy equal to the minimum energy. On the other hand it implies that the critical temperature for the ensemble $\mathcal{S}$ of walks is infinite. This is reminiscent of directed walks in $d \leqslant 2$, i.e. of a situation of highly correlated walks. Indeed in this case the number of walks is not macroscopic: with $\gamma = 0$ the equality in (17) implies that $\tilde{\Pi} = 1$.

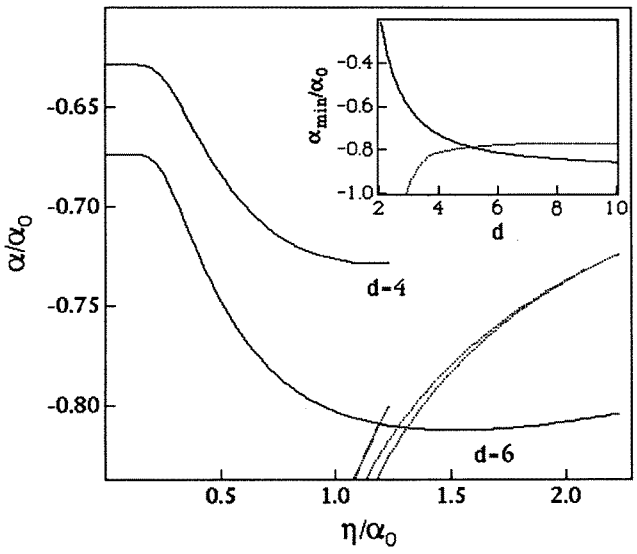

Figure 2. Behaviour of α as a function of η for $d = 4$ and $d = 6$. The data are scaled by the GS of independent walks $\alpha_0 = (2 \ln z)^{1/2}$. τ, which is fixed by saturating the inequality (14), is also shown as a dotted curve for the same dimensions. For $d = 4$, τ lies below α, whereas for $d = 6$ the two curves intersect. Inset: minimum GS energy $\alpha_{\min}$, and relative value of τ (dotted curve) as a function of d.

Finally note that there is a second dimension, d_1, above which the energy of the GS lies below the threshold τ. This does not however mean, as we shall discuss later, that the GS path lies mostly on sites with energies $\epsilon_i < \tau$. Also the value of d_1 is non-universal, in the sense that it depends on z and Π which are non-universal numbers.

4. Discussion

We have shown that directed polymers in a random medium with killing, within a range of the parameters, has a low temperature phase with an exponent $\omega = 0$ for the scaling behaviour of the sample to sample fluctuation of the GS energy with L, for $d > 2$. It is easy to realize that, even though we tried to build our model to eliminate mainly low energy walks, the set $\mathcal{S}$ almost surely misses the true GS, that of walks without killing. To see this let us consider the distribution of one site energy along the GS path of walks with killing. It is easy to see that, since the sum of the site energies along the path is constrained to have fluctuations of order 1, also each site energy has fluctuation of order one around its average value α (this has been shown also for DPRM on the Cayley tree [4]). Therefore the probability that the energy of a site exceeds the threshold τ is finite and the number of sites, on the GS path, with energy $\epsilon_i > \tau$ increases linearly with L. This means that the surviving probability is exponentially small for walks with energy αL. Since our calculation reveals that typically one of these walks survives the killing process, one concludes that their number is macroscopic, i.e. exponential in L. In other words the entropy of DPRM for energies αL is extensive and finite. This implies that our results explore only the high temperature phase of DPRM. The above result then yields an upper bound to the GS energy, i.e. the minimum $\alpha_{\min}$ obtained for a given dimension d. Note that, using the expression of the energy in the high temperature phase [21], this also yields an upper bound to the critical

temperature T_c [23]. The minimum $\alpha_{\min}$, as a function of dimension, is plotted in the inset of figure 2.

The fact that, on the GS path, each site energy can have fluctuations of the same order as the total energy, for $\omega = 0$, has striking consequences. The first is that, for Gaussian site energies, the largest site energy on the GS can be as large as $\sqrt{2 \ln L}$. Furthermore the site energies on the GS path have a negative pairwise correlation. To see this it is enough to note that $\langle (\delta \underline{E}_<)^2 \rangle = L \langle (\delta \underline{\epsilon}_i)^2 \rangle + L(L-1) \langle \delta \underline{\epsilon}_i \delta \underline{\epsilon}_j \rangle$ where i and j are two sites on the GS path. If $\langle (\delta \underline{\epsilon}_i)^2 \rangle \simeq 1$, in order for $\langle (\delta \underline{E}_<)^2 \rangle$ to be finite it is necessary that $\langle \delta \underline{\epsilon}_i \delta \underline{\epsilon}_j \rangle \sim -L^{-1}$.

One interesting feature of our model is that it allows us to relate the DPRM problem with directed percolation. This relation has already been exploited in various disguises by many authors [15, 23, 24]. In our case, one can define a cluster of directed percolation at concentration $\rho(\tau) = \int_{-\infty}^{\tau} \mathrm{d}x \, \phi(x)$, as the set of sites which can be connected to the origin by directed paths on sites with energies less than τ, i.e. with $p(\underline{\epsilon}_r) = 1$. As $\rho \to \rho_c \equiv \rho(\tau_c)$, the directed percolation threshold, this construction results eventually in an infinite percolating cluster.

Our previous argument, which suggests that the maximum site energy on the GS path can be as big as $\sqrt{\ln L}$, suggests that no matter what threshold τ one fixes, the GS path will sooner or later abandon the percolating cluster and minimize its energy by finding more favourable regions where the energy is small. This fact, by itself, implies that the GS path of DPRM will wander more than any path in a percolating cluster, and in particular more than paths within the incipient percolating cluster at τ_c. The wandering of the latter is described by an exponent $\zeta_{DP} = \nu_\perp / \nu_\parallel$, where $\nu_\perp$ and $\nu_\parallel$ are the exponents with which the correlation lengths in the space and time direction respectively diverge as $\tau \to \tau_c^-$ in directed percolation. Therefore one finds the exponent relation

$$\zeta \geqslant \zeta_{DP}. \tag{23}$$

Since $\zeta_{DP} = \frac{1}{2}$ only for $d \geqslant 4$, this implies that $\omega = 2\zeta - 1 = 0$ cannot apply to $d < 4$, i.e. $d_c \geqslant 4$. The inequality (23) was originally derived by Bouchaud and Georges [15].

The remainder of this section discusses how the results presented so far could be extended to an argument for $d_c = 4$.

It has been argued [24] that, as long as $\tau > \tau_c$, the statistical properties of the sub-set of directed polymers constrained in directed percolation clusters ($\eta = \infty$) are the same as those of the whole unconstrained set. The behaviour at $\tau = \tau_c$, $\eta = \infty$ belongs to a new universality class because the constraint imposed by directed percolation becomes too strong. To see this it is enough to observe that at $(\eta, \tau) = (\infty, \tau_c)$ the number of surviving walks is not macroscopic. This is not so if η is finite. This therefore suggests that

(i) the DPRM universality class extends at least down to the line $\tau = \tau_c$ in the (η, τ) plane (shown as the dashed region in figure 1).

Even though, as shown in figure 1, our results apply only in a region below the line $\tau = \tau_c$, it is interesting to consider ideally the situation with $\tau = \tau_c$ and $\eta \gg 1$ finite. Then in the set $\mathcal{S}$ there will be mainly walks which follows the percolating cluster, where there is no suppression because $p(\underline{\epsilon}_i) = 1$. Following arguments similar to those used in [26], one can observe that the intersection of the incipient percolating cluster with a 'constant time' hyperplane, can be described as a fractal of dimension $D = 2$, in $d \geqslant 4$. Two such clusters starting from different points will intersect, at time t, on a set of dimension $D_I = 2D - d = 4 - d$. If $D_I < 0$, which happens for $d > 4$, this result is to be interpreted probabilistically: the probability of having a non-empty intersection at time t is $\sim t^{D_I}$. For $d > 4$, directed percolation identifies infinite routes to infinity. One can exploit this geometrical information in our case by the following argument. Consider two different

realizations $\mathcal{S}_1$ and $\mathcal{S}_2$ of the killing process in the same disorder realization. Let $\bar{w}_1$ and $\bar{w}_2$ be the GS paths in the two realizations. Assumption (i) states that their behaviour is characterized by the same exponents which describe the low temperature phase of DPRM. In regions inside the percolation cluster, the two walks will experience a large effective attraction, because both will follow the best energy path. However, in regions outside the cluster, they will be scattered randomly by the killing process. As a result, even if $\bar{w}_1$ and $\bar{w}_2$ coincide inside the cluster, they will probably diverge outside it. They will, therefore, probably re-enter a region where $p_{\eta,\tau_c}(\underline{\epsilon}_i) = 1$, the percolating cluster, in different points. For $d < 4$, even in this case, the two paths have a large probability to meet again, because the percolating cluster itself is connected. Even if the the two paths join the cluster in two branches which are distinct at time t, the latter will sooner or later merge into a single branch. In contrast, in $d > 4$, these two branches will, most likely, never meet. It is therefore very likely that the two walks will also miss each other in $d > 4$. This argument suggests that

(ii) low energy walks (such as w_1 and w_2) will intersect in a number of points m which diverges with the length of the walks L in $d < 4$, but which is finite in $d > 4$.

We can use this estimate of the correlation between low energy walks, to get a rough evaluation of the exponent ω. This can be done under the assumption that

(iii) the distribution of the energies of low energy walks is Gaussian with a correlation coefficient $\langle \delta \underline{E}_w \delta \underline{E}_v \rangle \simeq m$.

The assumption of a constant correlation is not crucial and can be relaxed considerably [17]. The real assumption here is on the form of the distribution. The Gaussian form is, however, the only one for which results are available for any, even strong, correlation.

The minimum among a macroscopic number (exponential in L) of Gaussian variables with fluctuations $\langle (\delta \underline{E}_w)^2 \rangle \sim L$ and correlation $\langle \delta \underline{E}_w \delta \underline{E}_v \rangle \simeq m(L)$ is [17] a variable which has fluctuations proportional to $L^\omega \sim \sqrt{m(L)}$ (provided $m(L)L^{-2/3} \to 0$ as $L \to \infty$). Our previous argument, suggesting $m(L) \sim$ constant for $d > 4$, would therefore lead to the conclusion that $\omega = 0$ for $d > 4$, i.e. $d_c = 4$.

The assumptions (i), (ii) and (iii) involved in the above arguments, and therefore the conclusions, are clearly of a speculative nature at this stage. However, we think the argument provides a likely geometrical picture for the emergence of an upper critical dimension in the DPRM problem and we hope it can encourage further efforts in this direction.

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Appendix A. Independent walks

The limit distribution of the minimum $\underline{E}_< = \min(\underline{E}_1, \ldots, \underline{E}_N)$, among $N = z^L$ independent and identically distributed variables is obtained observing that if $\underline{E}_< \geqslant x$, then the $\underline{E}_w \geqslant x$, $\forall w$. The probability of the two events are therefore equal: $P_<(x) = 1 - [1 - P(x)]^N$ where $P(x) = \int^x p(E)\,\mathrm{d}E$ is the cumulative distribution of $\underline{E}_w$. This probability $P_<(x)$, which defines the distribution of $\underline{E}_<$, has a finite limit as $L \to \infty$ if one can find constants a_L and b_L such that $\underline{E}_< = a_L + b_L \underline{\Lambda}$, and the distribution of $\underline{\Lambda}$ is non-degenerate and independent of L. In order for

$$\lim_{L\to\infty} P_<(a_L + b_L x) = \lim_{L\to\infty} 1 - \exp\{z^L \ln[1 - P(a_L + b_L x)]\}$$

to be finite it is clear that one has to look for values of a_L and b_L for which $P(a_L + b_L x) \sim z^{-L}$. Defining

$$c(x) = \lim_{L\to\infty} z^L P(a_L + b_L x)$$

one finds easily that the limit distribution is

$$\lim_{L\to\infty} P_<(a_L + b_L x) = \exp[-c(x)]. \tag{A1}$$

Anticipating that $b_L = b$ is a constant, we can further simplify the evaluation of $c(x)$ by taking the derivative of both sides:

$$\frac{\mathrm{d}}{\mathrm{d}x}c(x) = b \lim_{L\to\infty} z^L p(a_L + bx) = b \lim_{L\to\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}[z\langle \mathrm{e}^{-\mathrm{i}kx}\rangle]^L \exp[\mathrm{i}k(a_L + bx)] \tag{A2}$$

where we used equation (2). Let us choose $a_L = \alpha L + \alpha' \ln L + \alpha''$, with α, α' and α'' constants. The integral in equation (A2) is dominated by the saddle point for $L \to \infty$. Collecting terms exponentially large in L, we find that the integral is dominated by the value $\mathrm{i}k = \gamma$ which is the solution of

$$\frac{\langle x\mathrm{e}^{-\gamma x}\rangle}{\langle \mathrm{e}^{-\gamma x}\rangle} = \alpha. \tag{A3}$$

In order to have a finite limit one has also to require that α is such that the coefficient of L in the exponential vanishes. This leads to

$$\alpha = -\frac{1}{\gamma}\ln(z\langle \mathrm{e}^{-\gamma x}\rangle). \tag{A4}$$

Note that equations (A3) and (A4) have, in general, two solutions. Indeed if we had asked about the maximum energy, instead of the minimum, we would have gone through exactly the same steps. Of these solutions we shall focus on the one with the smallest value of α. It is easy to see that, in this case, the dominant saddle point value corresponds to the largest value of γ.

If α and γ satisfy these two equations, the limit is dominated by the Gaussian fluctuations around the saddle point:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}c(x) &= b \lim_{L\to\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp\left[-\frac{L\sigma^2}{2}(k - \mathrm{i}\gamma)^2 + \mathrm{i}k(\alpha' \ln L + \alpha'' + bx)\right] \\ &= \lim_{L\to\infty} \frac{bL^{\gamma\alpha'}\mathrm{e}^{\gamma(\alpha''+bx)}}{\sqrt{2\pi\sigma^2 L}}\end{aligned}$$

where

$$\sigma^2 = -\frac{\mathrm{d}^2}{\mathrm{d}k^2}\ln\langle \mathrm{e}^{\mathrm{i}kx}\rangle\Big|_{k=\mathrm{i}\gamma}. \tag{A5}$$

In order to make the limit finite we need $2\gamma\alpha' = 1$. Choosing α'' and b in such a way that the limit equals e^x, we find equation (3) and the distribution $\lambda(x)$ of $\underline{\Lambda}$. Finally, equation (A5) yields equation (6).

Appendix B. Correlated walks

As before, the problem can be simplified by considering probability densities instead of cumulative distributions. Therefore one has to evaluate

$$\sum_{(w,v)\in \underline{C}_L} p_{(w,v)}(U_L; U_L) = |\underline{C}_L| \sum_{m=1}^{L} Q_m p(U_L; U_L|m) \tag{B1}$$

where $Q_m \sim \Pi^m$ is the probability that two walks intersect (Π is the probability for one intersection) and we have introduced the joint probability density $p(x; y|m)$ for the energies of two walks which intersect in m sites. This is given by

$$p(U_L; U_L|m) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \mathrm{e}^{\mathrm{i}(k+k')U_L} \langle \mathrm{e}^{-\mathrm{i}k\underline{\epsilon}} \rangle^{L-m} \langle \mathrm{e}^{-\mathrm{i}k'\underline{\epsilon}} \rangle^{L-m} \langle \mathrm{e}^{-\mathrm{i}(k+k')\underline{\epsilon}} \rangle^{m}. \tag{B2}$$

Collecting this information and using $\underline{C}_L \sim z^{2(L-\ell)}$, we find

$$z^{2(L-\ell)} \sum_{m=1}^{L} Q_m p(U_L; U_L|m) = L z^{-2\ell} \int_{1/L}^{1} \mathrm{d}\mu \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi} \mathrm{e}^{LG(\mathrm{i}k, \mathrm{i}k'|\mu)} \tag{B3}$$

where

$$G(x, y|\mu) = \alpha(x+y) + (1-\mu)(\ln\langle \mathrm{e}^{-x\underline{\epsilon}} \rangle + \ln\langle \mathrm{e}^{-y\underline{\epsilon}} \rangle) + \mu \ln\langle \mathrm{e}^{-(x+y)\underline{\epsilon}} \rangle + 2\ln z + \mu \ln \Pi. \tag{B4}$$

For each value of μ the integrals in k and k' are dominated by a saddle point. The location of the saddle point is given by $\partial_x G(x, y|\mu) = 0$ and $\partial_y G(x, y|m) = 0$. Because of the symmetry $x \leftrightarrow y$ there will be a solution $\zeta = x^* = y^*$. There could in general be others but we neglect this possibility (in all the cases examined there were no 'asymmetric' solutions). Therefore $\zeta = x^* = y^*$ is given by the equation

$$\partial_x G(x, \zeta|\mu)|_{x=\zeta} = \alpha - (1-\mu)\frac{\langle \underline{\epsilon} \mathrm{e}^{-\zeta\underline{\epsilon}} \rangle}{\langle \mathrm{e}^{-\zeta\underline{\epsilon}} \rangle} - \mu \frac{\langle \underline{\epsilon} \mathrm{e}^{-2\zeta\underline{\epsilon}} \rangle}{\langle \mathrm{e}^{-2\zeta\underline{\epsilon}} \rangle} = 0. \tag{B5}$$

It is easy to show that $\zeta(\mu)$ is a decreasing function of $\mu \in [0, 1]$. Moreover, $\zeta(\mu = 0) = \gamma$ and $\zeta(\mu = 1) = \gamma/2$, as is evident by comparing equations (B5) and (A4). For $\mu = 0$, $G(\zeta, \zeta|0) = 2(\alpha\zeta + \ln\langle \mathrm{e}^{-\zeta\underline{\epsilon}} \rangle + \ln z) = 0$ which is the equation which fixes γ. Finally, if

$$\frac{\mathrm{d}}{\mathrm{d}\mu} G(\zeta, \zeta|\mu) = \frac{\partial}{\partial\zeta} G(\zeta, \zeta|\mu) \frac{\mathrm{d}\zeta}{\mathrm{d}\mu} + \frac{\partial}{\partial\mu} G(\zeta, \zeta|\mu) = \ln\left[\frac{\Pi\langle \mathrm{e}^{-2\zeta\underline{\epsilon}} \rangle}{\langle \mathrm{e}^{-\zeta\underline{\epsilon}} \rangle^2} \right] \leqslant 0 \tag{B6}$$

one can conclude that the integral in μ is dominated by $\mu = 0$ and the limit vanishes provided $\ell \propto \ln L$ is big enough. One can verify that the quantity in equation (B6) is an increasing function of ζ, which is also necessary in order for Gaussian fluctuations around the saddle point to be stable. Therefore, since $\zeta(\mu) \downarrow \mu$, one only needs to verify equation (B6) for $\mu = 0$. This is just the inequality (8).

For example, in the case of normal site energies, for which $\alpha = -\sqrt{2\ln z}$, it is straightforward to find $G(x, y|\mu) = \alpha(x+y) + (x^2+y^2)/2 + \mu xy + \alpha^2 + \mu \ln \Pi$. Therefore there is only one saddle point solution to $\partial_x G(x, y|\mu) = \alpha + x + \mu y = 0$ with

$$x = y = \zeta \equiv -\frac{\alpha}{1+\mu} \tag{B7}$$

which is clearly a decreasing function of μ, and leads to

$$G(\zeta, \zeta|\mu) = \frac{\mu\alpha^2}{1+\mu} + \mu \ln \Pi. \tag{B8}$$

One can verify that $G(\zeta, \zeta|\mu) \leqslant 0$ if the condition (8) is matched, that is if $\Pi z^2 \leqslant 1$.

Appendix C. Thermodynamics of walks with killing

Taking the average of equation (10) over the variables $\underline{\kappa}_{w,t}$, leads to

$$Z_i(\beta) = \mathrm{e}^{-\beta\underline{\epsilon}_i}\, p(\underline{\epsilon}_i) \sum_{(j,i)} Z_j(\beta) \tag{C1}$$

where $i = (\boldsymbol{x}, L+1)$ and the sum runs on the z neighbours $(\boldsymbol{y}, L)$ of i. This equation states that a fraction $1 - p(\underline{\epsilon}_i)$ of all the walks arriving at site $i = (\boldsymbol{x}, L+1)$ is suppressed by killing at the $(L+1)$st step irrespective of the energy $\underline{E}_w(L)$ accumulated by the walks in the previous L steps. The effect of killing, here, is accounted for by observing that only a fraction $p(\underline{\epsilon}(\boldsymbol{x}, L+1))$ of the partition function up to step L 'survives'. Summing $Z_i(\beta)$ over all the sites $i = (\boldsymbol{x}, L)$ at 'time' L yields the partition function $Z(\beta, L)$. It is easy to find the expression for the free energy (density) in the high temperature phase

$$F_+(\beta) = -\frac{1}{\beta L} \ln\langle Z(\beta, L)\rangle = -\frac{\ln[z\langle p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle]}{\beta}. \tag{C2}$$

From this one can find the average energy

$$U_+(\beta) = \frac{\partial}{\partial\beta}[\beta F_+(\beta)] = \frac{\langle \underline{\epsilon}\, p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle}{\langle p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle}$$

and therefore the entropy

$$S(\beta) = \beta(U_+ - F_+) = \beta\frac{\langle \underline{\epsilon}\, p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle}{\langle p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle} - \ln[z\langle p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle]. \tag{C3}$$

Note that $S(\beta)$ vanishes at $\beta = \gamma$, where γ is defined in equation (13). One can show, following [21], that the expression (C2) yields the true free energy for any β such that

$$\lim_{L\to\infty} \frac{\langle Z^2(\beta, L)\rangle}{\langle Z(\beta, L)\rangle^2} < \infty. \tag{C4}$$

Indeed in this case one can conclude that $\langle \ln Z(\beta, L)\rangle = \ln\langle Z(\beta, L)\rangle$. It is easy to verify that equation (C4) holds if

$$\Pi\frac{\langle p^2(\underline{\epsilon})\mathrm{e}^{-2\beta\underline{\epsilon}}\rangle}{\langle p(\underline{\epsilon})\mathrm{e}^{-\beta\underline{\epsilon}}\rangle^2} \leqslant 1. \tag{C5}$$

Since the left-hand side is an increasing function of β, this equation holds for $\beta \leqslant \beta_c$ where β_c is defined as the temperature for which (C5) turns into an equality. If, as previously, the parameters in $p(x)$ are chosen in such a way as to recover the IID result and to minimize the GS energy, the equality in equation (17) implies that $\beta_c \geqslant \gamma$. Note that for $\beta_c < \gamma$, equation (C3) would yield a negative entropy. This is clearly a shortcoming of the approximation in equation (C1). Physically it means that the walks which dominate the partition sum $Z(\beta, L)$ survive with an exponentially small probability. In order to overcome these problems one should take the average over the variables $\underline{\kappa}$ of the logarithm of the partition function, by e.g. the replica trick, and not of the partition function itself as done in equation (C1).

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